Micro Notes DRAFT 1.0

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Contents

1 Basic Math

Result 1.1 $x \cdot y = \cos\theta ||x|| \cdot ||y||$

Remark θ is *acute* if $x \cdot y > 0$. It's *obtuse* if $x \cdot y < 0$. If the dot product is zero, the vectors are *orthogonal.*

Definition The *non-negative orthant* is $R^N_+ = \{x \in R^N : x \geq 0\}$

Definition The *positive orthant* is $R^N_{++} = \{x \in R^N : x >> 0\}$

Definition A property of a function is *ordinal* if it is preserved under any increasing transformation of that function

Remark For example, increasing functions remain increasing under increasing transformations so increasingness is an ordinal property of a function. Other examples are quasi-concavity and homotheticity.

Remark A property is *cardinal* if its not ordinal, like concavity, continuity and homogeneity

1.1 Linear Algebra

Definition The function $L : R^N \to R^M$ is a *linear function* if $\forall x, y \in R^N$ and $c \in R$:

$L(x+y) = L(x) + L(y)$

$L(cx) = cL(x)$

Theorem 1.2 *If $L : R^N \to R^M$ is linear then $\exists A \in M_{M\times M}$ s.t. $L(x) = Ax, \forall x \in R^N$.*

Proof See page 288 S&B[4]

Definition A function $Q : R^N \to R^M$ is quadratic if it has the form: $Q(x_1, \cdots, x_N) = a_{11}x_1x_1 + \cdots + a1Nx_1x_N + a_{21}x_2x_1 + \cdots + a_{2N}x_2x_N + \cdots + a_{N1}x_Nx_1 + \cdots + a_{NN}x_Nx_N = x'Mx$. Where $M = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \cdots & \cdots & \cdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}$

Definition $Q_{N\times N}$ is *positive (negative) definite* if $\forall x \neq 0, x'Qx > (<)0$. It's *positive (negative) semi-definite* if the inequality is weak. Otherwise, Q is *indefinite.*

Definition A *Principle sub-matrix of order* r of $A \in M_{n\times n}$ is the matrix obtained by deleting $n-r$ rows of A and the corresponding columns.

Definition The *leading principle sub-matrix of order* r, denoted rAr, is the principle sub-matrix of order r in which the last $n-r$ columns and rows are deleted.

Theorem 1.3 *Symmetric $A \in M_{n\times n}$ is positive definite iff $|rAr| > 0, \forall 0 < r \leq n$.*

Proof See page 393 S&B[4]

Theorem 1.4 *Symmetric $A \in M_{n\times n}$ is negative definite iff $(-1)^r|rAr| > 0, \forall 0 < r \leq n$.*

Proof ?

Result 1.5 $M^{-1} = 1/detM \cdot$ *adj* M

Result 1.6 $\frac{\partial(x\cdot Mx)}{\partial x} = 2Mx$

1.2 Topology

Theorem 1.7 *S is open in $X \subset R^N$ iff $\exists A \subset R^N$, open s.t. $S = X \cap A$.*

Proof Assume there exists such an A. Let $x \in S$, in particular this means its also in A. So, given A is open, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset A$. This implies $B_\epsilon(x) \cap X \subset S$ which implies S is open. ∎

Definition $S \subset R^N$ is *convex* if $x, y \in S, \theta \in [0,1]$ then $(\theta x + (1-\theta)y) \in S$.

Definition For the convex set S, $f : S \to R$ is *concave* if $f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y), \forall x, y \in S, \forall \theta \in [0,1]$. This is *strict* if the inequality is strict and the $\theta \in (0,1)$. For convexity just reverse the inequality.

Theorem 1.8 *f is convex iff $-f$ is concave*

Theorem 1.9 *$f : S \to R \in C^1$ is concave iff $f(x+v) \leq f(x) + \nabla f(x) \cdot v, \forall v$*

Definition f is *quasi-concave* if all its upper contour sets are convex, i.e. $f(x) \geq z, f(y) \geq z, \theta \in [0,1] \Rightarrow f(\theta x + (1-\theta)y) \geq z$.

Theorem 1.10 *f is quasi-concave iff $f(\theta x + (1-\theta)y) \geq \min\{f(x), f(y)\}$*

Theorem 1.11 *$f \in C^1$ is quasi-concave iff $f(x) \geq f(y) \Rightarrow \nabla f(x) \cdot (y-x) \geq 0$.*

Theorem 1.12 *If $f(y) \geq f(x), x \neq y \Rightarrow \nabla f(x) \cdot (y-x) > 0$ then f is strictly quasi-concave.*

Theorem 1.13 *Let $f : S \to R \in C^2$ then*

f is concave iff $D^2 f(x)$ is negative semi-definite

If $D^2 f(x)$ is negative definite, $\forall x \in S$ then f is strictly concave.

Theorem 1.14 *$f : A \to R \in C^2$ is quasi-concave iff $D^2 f(x)$ is negative semi-definite on the subspace orthogonal to the gradient, e.g. if v solves $\nabla f(x) \cdot v = 0$ then $v' D^2 f(x) v \leq 0$.*

Theorem 1.15 *If $\forall x \in A$, $D^2 f(x)$ is negative definite on the subspace orthogonal to the gradient then f is strictly quasi-concave.*

Remark *Quasi-convex* is when all the lower contour sets are convex. Thus, all the results for quasi-concavity above can be rewritten for quasi-convexity.

1.3 Calculus

Definition If f is differentiable at x^*, then the linear function $Lf(x^*) : R \to R$, $Lf(x^*)(v) = f'(x^*)v$ is a *first order approximation* to the function $\phi : R \to R, \phi(v) = f(x^* + v) - f(x^*)$ at $v = 0$.

Result 1.16 *Implicit function theorem: If* $f(x, y) = c$*, a constant, then* $\frac{\partial y}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ *(if its well-defined).*

Remark The tangent vector of $f(x, y)$ at (x^*, y^*) is $(1, -\frac{\partial f/\partial x}{\partial f/\partial y})$. It can be verified that this vector is orthogonal to the gradient vector (take the dot product, its zero).

Definition $f : S \to R, S \subset R^N$ s.t. $x \in S \Rightarrow ax \in S, \forall a \in R^+$, is *homogeneous of degree* r, denoted $r - homo$, if $f(tx) = t^r f(x), \forall x \in S, \forall t > 0$

Theorem 1.17 *If* $f : S \to R$ *is* $r - homo$ *and differentiable, then* $\partial f/\partial x_j(x)$ *is* $(r - 1) - homo$*.*

Proof Let $\phi : S \times R_{++} \to R, \phi(x, t) = f(tx) - t^r f(x)$. $\frac{\partial \phi}{\partial x_j}(tx) = \frac{\partial f(tx)}{\partial x_j}\frac{\partial (tx_j)}{\partial x_j} - t^r \frac{\partial f(x)}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} \cdot t - t^r \frac{\partial f(tx)}{\partial x_j}$. As f is $r - homo$, $\phi(x, t) = 0$. Thus, $t\frac{\partial f(tx)}{\partial x_j} = t^r \frac{\partial f(x)}{\partial x_j} \Rightarrow \frac{\partial f(tx)}{\partial x_j} = t^{r-1}\frac{\partial f(x)}{\partial x_j}$. ▮

Theorem 1.18 Euler's Theorem *If* $f : S \to R \in C^1$ *is* $r - homo$ *then* $Df(x) \cdot x = rf(x)$

Proof Let ϕ be as above 1.17. $\frac{\partial \phi}{\partial t} = \sum \frac{\partial f}{\partial x_j} \cdot \frac{\partial (tx_j)}{\partial t} - rt^{r-1}f(x) = \sum \frac{\partial f}{\partial x_j} \cdot x_j - rt^{r-1}f(x) = 0$. Take $t = 1$, then $\sum \frac{\partial f}{\partial x_j} \cdot x_j = rf(x)$. ▮

Definition $g : S \to R$ is *homothetic* if $g(x) = h(f(x))$ for $f : S \to R$, $1 - homo$ and $h : f(S) \to R$ and h is an increasing function.

Remark Homothetic functions share contour curves with their homogeneous function.

border matrices? see S&B page 391 and page 527 [4]

Theorem 1.19 *For* $f : A \to R$*,* A *convex s.t.* $x \in A$ *and* $t > 0 \Rightarrow tx \in A$*, if* $f(x) > 0, \forall x \in A$*,* f *is quasi-concave and* f *is* $1 - homo$ *then* f *is concave.*

Theorem 1.20 f *is concave iff* f*'s hypo-graph is convex.*

1.4 Optimization

For everything in this section, $f : A \times Q \to R, A, Q \subset R^N$.

Definition Given $q \in Q$, $\bar{x}$ is a *local maximizer* of f if $\exists B_\epsilon(\bar{x})$ s.t. $x \in B_\epsilon(\bar{x}) \cap A$ implies $f(x) \leq f(\bar{x})$.

Remark Epsilon balls, like $B_\epsilon(\bar{x})$, are intervals in the real line.

Result 1.21 *If $f \in C^1$ and $A \times Q$ is compact (bounded and closed), then $\exists x \in A \times Q$ s.t. $f(x) \geq f(y), \forall y \in A \times Q$.*

Definition The *solution correspondence*, $\xi : Q^* \to R^N, \xi(q) = \{x \in A : x$ solves $\max_x f(x,q)\}$ where $Q^* = \{q : \max_x f(x,q) \neq \{\}\}$

Definition The *value function*, $v : Q^* \to R, v(q) = f(\bar{x})$ for $\bar{x} \in \arg\max_x f(x,q)$.

1.4.1 Constrained Optimization

Let $g : A \to R^M$,

$$\max_{x \in A, g(x)=c} f(x) \tag{1}$$

Definition The *constraint set*, $C = \{x \in A : g(x) = c\}$

Definition $\bar{x}$ *solves* the problem 1 if $\bar{x} \in C$ and $f(\bar{x}) \geq f(x), \forall x \in C$.

Definition $\bar{x}$ *locally solves* the problem 1 if $\bar{x} \in C$ and $\exists B_\epsilon(\bar{x})$ s.t. $f(\bar{x}) \geq f(x), \forall x \in C \cap B_\epsilon(\bar{x})$.

Theorem 1.22 *Let $A \subset R^N$ be open and $f, g \in C^1, \bar{x} \in \arg 1$ and $Dg(\bar{x})$ be of full rank. Then $\exists \bar{\lambda} \in R^M$ s.t. $\nabla f(\bar{x}) = \sum_{m=1}^{M} \lambda_m \nabla g_m(\bar{x}) = \lambda \cdot Dg(\bar{x})$.*

Remark At the constrained maximum, the tangent of the constraint function corresponds to the tangent of the objective function. For some x, this theorem is a necessary condition for it to be a maximum (not sufficient however). This implies the LaGrangian method for finding potential maxima. Let $L : A \times R^M \to R, L(x, \lambda) = f(x) - \lambda \cdot (g(x) - c)$ then the theorem implies that for maxima $\nabla L(x, \lambda) = 0$

Remark Economist don't usually check second order conditions for sufficiency of a maximand (i.e. the objective needs to be negative semi-definite on the constrained space). It's usually assumed that the objective and the constraints are of the 'right shape' for a maximum (otherwise the LaGrange method could find minima).

Remark The above problem only had 'binding' constraints (i.e. equality). Most economics problems have much more loose constraints (e.g. expenditure is *less than* the budget). We extend the problem to accommodate these sorts of constraints.

Let $h : A \times Q \to R^K$, given $q \in Q$,

$$\max_{x \in A, g(x)=b, h(x) \le c} f(x,q) \tag{2}$$

Theorem 1.23 *Kuhn-Tucker: In respect to 2, if* $\bar{x} \in C$ *is a maximizer and* $\{\nabla g_m : \forall m\} \cup \{\nabla h_k : \forall k\}$ *are linearly independent then*

1. $\forall g_m, \exists \lambda_m \in R$ *and* $\forall h_k, \exists \lambda_k$ *s.t.*
2. $\nabla f(\bar{x}) = \sum_{m=1}^{M} \lambda_m \nabla g_m + \sum_{k=1}^{K} \lambda_{M+k} \nabla h_k$
3. $\lambda_{M+k}(h_k(\bar{x}) - c_k) = 0$

Remark 123 are called the Kuhn-Tucker conditions. As before, these are characteristics of maximizers so if a point satisfies these conditions it could be a maximizer.

Remark 1 can be rewritten as $\nabla L(\bar{x}, \lambda) = 0$

Result 1.24 *It is sufficient for a point to be a maximum if the Kuhn-Tucker conditions hold (1,2 and 3), the constraint functions* $(g(x), h(x))$ *are quasi-convex (1.2) and* $\nabla f(\bar{x}) \cdot (x - \bar{x}) > 0, \forall x$ *s.t.* $f(x) > f(\bar{x})$.

Remark I think the last two conditions of the above result are more general versions of a condition that the constraints are concave.

1.4.2 Envelope Theorem

Theorem 1.25 *Let* $v(q)$ *be the value function for the problem 2, then* $\forall q_s$ *(parameters)* $\frac{\partial v(q)}{\partial q_s} = \frac{\partial L(x,q,\lambda)}{\partial q_s}$.

1.5 Functional forms

1-homo, x solves ump(p,1) implies wx solves ump(p,w).

quasi-linearity, consumer surplus

Definition Demand is of the gorman form if it can be written such that wealth and prices are separable (i.e. $x^i_j(p, w_i) = \alpha^i_j(p) + \beta^i_j(p)w_i$).

other utility forms, CES, Stone-Geary, linear, Leontief... FOC

1.6 Game Theory

A *game* is (1) a set of players $N = \{1, \cdots, n\}, n > 1, \forall i \in N, \exists S_i$ which is player i's set of strategies. (2) $S = S_1 \times \cdots \times S_n$ is the *strategy profile* for the game and W is the *set of outcomes.* (3) $f : S \to W$ is the *outcome function* and (4) $\forall i \in N, \exists \succ_i$ which are player i's preferences over W.[1]

Definition A player's *utility function*, $u_i : W \to R$.

Definition The *payoff function*, $\Pi_i(s) = u_i(f(s)), s \in S$.

Definition For player i, $s_i \in S_i$ is a *strictly dominant strategy* if $\forall s'_i \neq s_i, f(s_i, s_{-i}) \succ_i f(s'_i, s_{-i}), \forall s_{-i} \in S_{-i}$ where s_{-i} indicates a strategy profile for all other players.

Remark In other words, a strategy is strictly dominant if the outcome of pursuing that strategy is best no matter what the other players choose.

Definition For player i and $a, b \in S_i$, *a weakly dominates b* if $f(a, s_{-i}) \succeq_i f(b, s_{-i}), \forall s_{-i} \in S_{-i}$ and $\exists s'_{-i} \in S_{-i}$ s.t. $f(a, s'_{-i}) \succ_i f(b, s'_{-i})$.

Definition For player i and $a, b \in S_i$, a is *equivalent* to b if $f(a, s_{-i}) \sim_i f(b, s_{-i}), \forall s_{-i} \in S_{-i}$,

Remark Of course, all the $f(a, s_{-i}) \succ_i f(b, s_{-i})$ could be replaced with their utility equivalents... $\Pi_i(a, s_{-i}) > \Pi_i(b, s_{-i})$.

Definition Dominant Strategy Equilibrium: an outcome where each player chooses a weakly dominant strategy.

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[1] For a good introductory game thoery text see [1]

Definition $s \in S$ is a *Nash Equilibrium* if $\forall i \in N, \Pi_i(s_i, s_{-i}) \geq \Pi_i(s_i', s_{-i}), \forall s_i' \in S_i$.

Remark A Nash Equilibrium is a situation where each player doesn't regret their decision of strategy given the strategies of the other players.

-iterated elimination
-backwards induction
-perfect recall
-pure strategies
-mixed strategies
-Nash theorems

Theorem 1.26 *If* $\forall i \in N, S_i$ *is convex and compact,* Π_i *is continuous and concave in* S_i *then* $\exists q^* \in S = S_1 \times \cdots \times S_n$*, where* $S_i \subset R_+$*, is a Nash equilibrium.*

Remark Given finite time, demand cannot be infinite (all consumption takes some time) so demand has a finite bound, i.e. $S_i = [0, \bar{Q}]$, a convex and compact set. Given continuous demand and continuous cost, Π_i is continuous. Further, in the case of Cournot demand 7.4 $\Pi_i = p(q)q_i - c_i(q_i)$, so $\frac{\partial^2 \Pi_i}{\partial q_i^2} = \frac{\partial p}{\partial q_i} + \frac{\partial p}{\partial q_i} + \frac{\partial^2 p}{\partial q_i^2} q_i - \frac{\partial^2 c_i}{\partial q_i^2} \leq 0$ if demand is decreasing and concave and marginal costs are non-increasing. These assumptions on demand and cost are pretty standard economic assumptions, so given a non-finite strategy set we're guaranteed a pure strategy Nash Equilibrium. Finite strategy sets are not convex, but the use of mixed strategies serves as a means to convexify the strategy sets. Thus, this theorem holds if you consider mixed strategies and finite strategy sets.

Theorem 1.27 *If* $q^* \in S_i = [0, \bar{Q}]^N$ *is a Nash Equilibrium,* $\exists \tilde{q}$ *s.t.* $\forall i \in N, \tilde{q}_i < q_i^*$ *and* $\Pi_i(\tilde{q}) > \Pi_i(q^*)$*.*

Proof $q_i \in intS_i$ because $\sum q_i^* < \bar{Q}$ (recall $\bar{Q}$ is the bound for *total* demand). Because we have an interior solution, $\frac{\partial \Pi_i}{\partial q_i}(q^*) = 0$. Now choose $\bar{q} = q^* - \epsilon$ then take the first order Taylor expansion of $\Pi_i(q)$ evaluated at $\bar{q}$. $\Pi_i(\bar{q}) - \Pi_i(q^*) \approx \sum_{j \neq i} \frac{\partial \Pi_j}{\partial q_j}(q^*)(\bar{q} - q^*)$. The right hand side is positive — $\frac{\partial \Pi_i}{\partial q_j} = q_i \frac{\partial p}{\partial q_j} \Rightarrow \frac{\partial \Pi_i}{\partial q_j}(q^*) < 0$ — so $\Pi_i(\bar{q}) > \Pi_i(q^*)$. ∎

Remark This proof is in the context of Cournot demand 7.4. The general proof just requires the conditions on the demand functions that Cournot competition implies.

Remark Anti-trust laws then serve the function of preventing firms to collude to this more profitable, yet social not optimal, equilibrium.

Remark Is it easy to pick an $\epsilon(\Pi_i(q^*))$ such that the error term of the Taylor expansion isn't too big?

Remark summary of solution concepts?

2 Decision Theory

Let χ be the set of all alternatives. When we're talking about a world with certainty, $\chi \in R$. With uncertainty, χ is a set of probability distributions.

2.1 Choice

Definition A *choice structure* is a pair (β, C) where $\beta \neq \emptyset$ and $B \in \beta$ implies $B \subset \chi$ and $C : \beta \Longrightarrow \chi, C(B) \subset B, \forall B \in \beta$.

Definition The Weak Axiom of Revealed Preference (WARP): (β, C) satisfies WARP if $x, y \in B \in \beta$ and $x \in C(B)$ implies $\forall B'$ s.t. $x, y \in B'$ and $y \in C(B')$ then $x \in C(B')$, too.

Remark WARP says that if x is ever seen to be chosen with y, then it can never be the case that when both options are available only one of the two is chosen.

Definition The *Walrasian demand correspondence*, $x(p, w)$, selects a consumption bundle for each price-wealth pair.

Remark If this correspondence is single-valued (i.e. if for every price-wealth pair only one consumption bundle is chosen) then we call it the *Walrasian demand function.*

Definition $x(p, w)$ satisfies *Walras' Law* if for any given p and w, $\forall x \in x(p, w), p \cdot x = w$.

Definition A *Slutsky compensated price-wealth change* is one where if $\Delta p = p' - p$ then $\Delta x = x(p', p' \cdot x(p,w)) - x(p,w)$.

Theorem 2.1 *The Compensated Law of Demand: If Walras' Law holds for* $x(p,w)$ *then WARP 2.1 is satisfied if and only if* $\Delta p \cdot \Delta x \leq 0$ *for all Slutsky compensated price-wealth changes.*

Proof ($\Rightarrow$) $\Delta p \cdot \Delta x = (p' - p)(x(p', p' \cdot x(p,w)) - x(p,w)) = p'(x(p', p' \cdot x(p,w)) - x(p,w)) - p(x(p', p' \cdot x(p,w)) - x(p,w))$. Walras' Law implies $p' \cdot x(p', p' \cdot x(p,w)) = w'$ but $w' = p' \cdot x(p,w)$ by construction of the compensated wealth change. Thus, the first term is zero.

Now, $x(p,w)$ is affordable with $(p', p' \cdot x(p,w))$. But that bundle isn't chosen so $x(p', p' \cdot x(p,w))$ is revealed preferred to $x(p,w)$. Given, $x(p,w)$ is chosen when (p,w) then by WARP $x(p', p' \cdot x(p,w))$ isn't available under this price-wealth combination. This must mean that its more expensive, $p \cdot x(p', p' \cdot x(p,w)) > w = p \cdot x(p,w)$ where the last equality follows from Walras' Law. Thus, $\Delta p \cdot \Delta x < 0$. Equality follows if the bundle doesn't change after the Slutsky compensation.

($\Leftarrow$) see M-C [3] page 31

Remark If you remove the wealth effects of a price change, demand always slopes down.

Definition The *Slutsky Matrix*, $S(p,w) = D_p x(p,w) + D_w x(p,w) \cdot x(p,w)$

Result 2.2 *If* $x(p,w) \in C^1$*, 0-homo, WARP and satisfies Walras' Law. Then its Slutsky matrix is negative semi-definite.*

2.1.1 Comparative Statics

Definition Good j is considered *normal* (*inferior*)if $\frac{\partial x_j}{\partial w} > 0$ (< 0).

Definition For a given price level, the function/graph $x_j(w)$ is called the *Engel curve.*

Definition *Budget share* of good j is $B_j(p,w) = \frac{p_j \cdot x_j(p,w)}{w}$

Definition For all prices except for good j set and a given wealth, $x(p_j)$, is called the offer curve.

Theorem 2.3 *If the Walrasian demand function, $x(p,w)$, is 0-homo1.3, then*

$$D_p x(p,w) \cdot p + D_w x(p,w) \cdot w = 0 \tag{3}$$

Proof By Euler's Theorem 1.18, $D_p x(p,w) \cdot p = D_w x(p,w) \cdot w = 0$ because $x(p,w)$ is 0-homo. $\blacksquare$

Theorem 2.4 *If $x(p,w)$ satisfies Walras' Law then*

$$p \cdot D_p x(p,w) + x(p,w) = 0 \tag{4}$$

$$p \cdot D_w x(p,w) = 1 \tag{5}$$

Proof Because of Walras' Law, $p \cdot x(p,w) = w$. Differentiate by price and wealth, respectively to get the results. $\blacksquare$

Remark The above results are called Cournot 4 and Engel 5 aggregation respectively.

Remark Total expenditure doesn't change with price changes and expenditure must change as much as wealth when it changes.

2.1.2 Elasticity

Elasticity is the percent change in the variable of interest relative to a percent change in the parameter (e.g. wealth or price in this case). If quantity demanded changes 20% when price increases 10% then elasticity is 2 and the good is said to be elastic in price. On the other hand if quantity demanded only changes 5% then elasticity is .5 and the good is said to be inelastic.

Definition $\epsilon_{j,w}(p,w) = \frac{\partial x_j}{\partial w} \cdot \frac{w}{x_j(p,w)} = \frac{\frac{\partial x_j}{\partial w}}{\frac{x_j(p,w)}{w}}$ is the *wealth elasticity of good j*

Definition $\epsilon_{j,p_k}(p,w) = \frac{\partial x_j}{\partial p_k} \cdot \frac{p_k}{x_j(p,w)} = \frac{\frac{\partial x_j}{\partial p_j}}{\frac{x_j(p,w)}{p_j}}$ is the *price elasticity of good j with respect of good k*

Result 2.5 *Cournot and Euler aggregation can be expressed as relationships between wealth and price elasticities and budget shares,*

$$B(p,w) \cdot \epsilon_k(p,w) + B_k(p,w) = 0 \tag{6}$$

$$B(p,w) \cdot \epsilon_w(p,w) = 1 \tag{7}$$

2.2 Preference and rational preferences

Definition The preference relation (or binary operator) $\succeq$ on χ is *complete* if $\forall x, y \in \chi, x \succeq y$ or $y \succeq x$.

Definition $\succeq$ is *transitive* if $\forall x, y, z \in \chi, x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$.

Definition $\succeq$ is *rational* if it is complete and transitive

Example Condercet Voting Paradox

Let $\chi = \{x, y, z\}$ and let three people have the following rational preferences,

1. $x \succeq y, y \succeq z$
2. $y \succeq z, z \succeq x$
3. $z \succeq x, x \succeq y$

Let *social preferences* be such that $x \succeq y$ if more people prefer x to y. It's easy to see that the social preferences aren't transitive.

Definition $\succeq$ is *monotone* if $x >> y$ implies $x \succ y$. Its *strongly monotone* if $x \geq y$ and $x \neq y$ implies $x \succ y$.

Definition $\succeq$ is *locally non-satiated* if $\forall x \in \chi$ and $\forall \epsilon, \exists x' \in B_\epsilon(x)$ s.t. $x' \succ x$.

Theorem 2.6 *Strong monotonicity implies monotonicity implies local non-satiation.*

Proof Assume $\succeq$ is strongly monotone. Then $x >> y$ implies $x \geq y$ so $x \succ y$. Thus, $\succeq$ is monotone. Because its monotone, for any small distance away from x such that at least one of the components is greater, $x' \succ x$ so $\succeq$ is locally non-satiated. ▮

Definition $\succeq$ is *convex* if $\forall x, \{y \in \chi : y \succeq x\}$ is convex.

Remark This generalizes the concept of diminishing marginal rate of substitution between two goods. An assumption of convexity is an expression of agents desire for diversity. If x and y are two bundles that the agent is indifferent between then some combination of those bundles wouldn't be seen to be worse than either bundle.

Definition The *utility function* $u : \chi \to R$ *represents* the preference relations $\succeq$ if $u(x) \geq u(y)$ when $x \succeq y$.

Definition Lexographical preferences, for $\chi = R^2$, are such that $x \succeq y$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$.

Remark Lexographical preferences are rational but they can't be represented by a utility function.

Definition $\succeq$ is continuous if $x^n \succeq y^n, \forall n$ and $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$ and $x \succeq y$.

Theorem 2.7 *If $\succeq$ is continuous then there is a continuous utility function that represents $\succeq$.*

Proof This proof is just shows that a utility function exists that represents $\succeq$. For proof of continuity see [3] page 48.

Let $1 \in \chi$ be the unit vector. The indifference curve that contains any $x \in \chi$ must cross the ray through 1. So for some $\alpha, \alpha 1$ is indifferent to x (this requires $\succeq$ to be monotone, too). This α is a function of x and we take this for our utility function. Does $\alpha(x)$ represent $\succeq$?

Suppose $\alpha(x) \geq \alpha(y)$ then by monotonicity, $\alpha(x) \cdot 1 \succeq \alpha(y) \cdot 1$. Since $x \sim \alpha(x) \cdot 1$ and $y \sim \alpha(y) \cdot 1$, by rationality $x \succeq y$. Now suppose $x \succeq y$. We know $x \sim \alpha(x) \cdot 1$ and $y \sim \alpha(y) \cdot 1$ then $\alpha(x) > \alpha(y)$. ∎

Remark Why are lexographical preferences discontinuous then? Assuming they're continuous is equivalent to assuming there's a one-to-one mapping from the rationals to the reals which is impossible (the reals are bigger than the rationals).

Theorem 2.8 *If $\succeq$ can be represented by a utility function then it is rational.*

Proof Let u be the utility function that represents $\succeq$ and let $x, y, z \in \chi$ s.t. $u(x) \geq u(y) \geq u(z)$. By the transitivity and completeness of the reals, the preference relation is rational. ∎

Definition $\succeq$ *generates* the choice structure (β, C^*) where $C^*(B, \succeq) = \{x \in B : x \succeq y, \forall y \in B\}$.

Result 2.9 *The choice correspondence C^* generated by a rational $\succeq$ satisfies WARP.*

2.3 Utility maximization

For bundles of goods $x \in R^L$, let the utility maximization problem (UMP) be, given wealth w and prices p,

$$\max_{x \geq 0, p \cdot x \leq w} u(x) \tag{8}$$

Theorem 2.10 *If* $p >> 0$ *and* $\{x \in R^L_+ : x \geq 0, p \cdot x \leq w\} \neq \emptyset$ *then there exists a solution to the maximization problem.*

Proof The constraint set is closed and bounded (thus compact) — its a closed triangle in the positive orthant — and the utility function is continuous so a maximizer exists by 1.21. ▮

Remark Let the Walrasian demand correspondence — the set of solutions to the UMP — be, $\xi : R^L_{++} \times R_+ \Rightarrow \chi$ and, if single valued, call the Walrasian demand function $x : R^L_{++} \times R_+ \rightarrow \chi$

Theorem 2.11 *If* $u(x) \in C^0$ *and* $\chi = R^L_+$ *and* $\succeq$ *is locally non-satiated then*

1. ξ *is 0-homo*
2. *Walras' law is satisfied*
3. *If* $\succeq$ *is convex, then* $\xi(p, w)$ *is convex. Further, if* $\succeq$ *is strictly convex then* $\xi(p, w)$ *is single valued... i.e.* $x(p, w)$ *is well defined.*

Proof For 1, the budget set is unchanged when price and wealth are each changed by the same amount. For part 2, assume Walras' Law doesn't hold so $p \cdot \bar{x} < w, \bar{x} \in \xi(p, w)$. By local non-satiation, there must be a y near x such that y is feasible and $y \succ x$. This of course contradicts $x \in \xi$, so Walras' Law must hold. For part 3, take $x, y \in \xi(p, w)$. Let $u(x) = u(y) = u^*$ and $z = \alpha x + (1 - \alpha) y, \alpha \in [0, 1]$ then we know that $u(z) \geq u^*$ by quasi-concavity. But since z is feasible (the budget set is convex), $u(z) > u^*$ would contradict utility maximization. Thus, $u(z) = u^*$ and so $z \in \xi(p, w)$. ▮

Theorem 2.12 *If* $u(x) \in C^0$, $\succeq$ *is locally non-satiated and* $\chi = R^L_+$ *then the value function* $v(p, w) = u(x \in \xi(p, w))$ *is*

1. *0-homo*

2. *increasing in w, decreasing in p and non-increasing in $p_l, \forall l$*

3. *quasi-convex*

4. *continuous*

Proof The proof of 1 follows from the same result for ξ. For part 2, observe what happens to the budget set and then use an argument similar to the one for Walras' Law in the previous result. 4 follows from the continuity of $u(x)$ proved in the appendix A of chapter 3 in [3].

Now for the proof of quasi-convexity. We have to show that the lower contour sets are convex. Take $v(p,w) \leq \hat{v}, v(p',w') \leq \hat{v}$ and $(p'',w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$. For (p'',w'') and any feasible x, is $u(x) \leq \hat{v}$?

Assuming $p'' \cdot x \leq w''$, then $(\alpha p + (1-\alpha)p') \cdot x = \alpha p \cdot x + (1-\alpha)p' \cdot x \leq \alpha w + (1-\alpha)w'$. x, then, is either feasible with (p,w) or (p',w'). In either case, $u(x) \leq \hat{v}$.

Lemma 2.13 *The LaGrangian multiplier associated with the budget constraint is the marginal value of wealth.*

Proof $\frac{\partial v(p,w)}{\partial w} = \frac{\partial u(x(p,w))}{\partial w} \Rightarrow \nabla u(x) \cdot D_w x(p,w) = \lambda \cdot p \cdot D_w x(p,w)$ by the first order conditions of the UMP. By Engel aggregation, $p \cdot D_w x(p,w) = 1$ so $\frac{\partial v(p,w)}{\partial w} = \lambda$. ■

Theorem 2.14 *Roy's identity:* $x_j(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_j}}{\frac{\partial v(p,w)}{\partial w}}$.

Proof By the envelope theorem, $\frac{\partial v(p,w)}{\partial p_j} = -\lambda x_j$ but from the above result 2.13, $\frac{\partial v(p,w)}{\partial w} = \lambda$. So, substituting for λ we get our result after rearranging. ■

Remark This result also follows from the duality theorem 2.20

2.4 Expenditure minimization

For bundles of goods $x \in R^L$, let the expenditure minimization problem (EMP) be, given minimum utility level u and prices p,

$$\min_{x \geq 0, u(x) \geq u} p \cdot x \tag{9}$$

Below are the results for the EMP that correspond to 2.11 and 2.12 for UMP. Let $e(p,u)$, called the expenditure function, be the problem's value function. Let $\eta(p,u)$ and $h(p,u)$ be the corresponding solution correspondence and function and call them Hicksian demand.

Result 2.15 *If* $u(x) \in C^0$, $\chi = R^L_+$, $p >> 0$ *and* $\succeq$ *is locally non-satiated then*

1. η *is 0-homo in* p
2. *No excess utility, i.e.* $x \in \eta(p,u)$ *implies* $u(x) = u$.
3. *If* $\succeq$ *is convex, then* $\eta(p,u)$ *is convex. Further, if* $\succeq$ *is strictly convex then* $\eta(p,u)$ *is single valued... i.e.* $h(p,u)$ *is well defined.*

Result 2.16 *If* $u(x) \in C^0$, $\succeq$ *is locally non-satiated and* $\chi = R^L_+$ *then the expenditure function* $e(p,u)$ *is*

1. *1-homo in* p
2. *increasing in* u, *increasing in* p *and non-decreasing in* $p_l, \forall l$
3. *concave*
4. *continuous*

Theorem 2.17 *Hicksian Compensated Law of Demand: If* $u(x) \in C^0$, $\succeq$ *is locally non-satiated,* $p >> 0$ *and* $x'' \in \eta(p'',u), x' \in \eta(p',u)$, *then* $(p'' - p') \cdot (x'' - x') \leq 0$.

Proof $p'' \cdot x'' \leq p'' \cdot x'$ and $p' \cdot x' \leq p' \cdot x''$. Thus, $(p'' - p') \cdot x'' \leq (p'' - p') \cdot x'$ which is our result. ▮

Theorem 2.18 *Given* $\succeq$ *is strictly convex,* $u(x) \in C^0$ *and* $\chi = R^L_+$, $\nabla_p e(p,u) = h(p,u)$.

Proof $e(p,u) = p \cdot h(p,u)$ so $\nabla_p e(p,u) = h(p,u) + p \cdot D_p h(p,u)$. From the first order conditions, $p = \lambda \nabla u(h(p,u))$. Thus, $\nabla_p e(p,u) = h(p,u) + \lambda \nabla u(h(p,u)) \cdot D_p h(p,u)$. Because there's no excess utility $u(h(p,u)) = u$, a constant. Thus, its derivative is zero and we have our result. ▮

Remark This result could be proved with the forthcoming duality theorem 2.20 and the envelope theorem1.25. See [3] page 68.

Result 2.19 *Given $\succeq$ is strictly convex, $u(x) \in C^1$ and $\chi = R^L_+$, the following are true:*

1. $D_p h(p,u) = D^2_p e(p,u)$
2. $D_p h(p,u)$ *is a negative semi-definite, symmetric matrix*
3. $D_p h(p,u)p = 0$

2.5 Relating demand, implicit utility and expenditure

Theorem 2.20 *Duality Theorem: If $u(x) \in C^0$, $\succeq$ is locally non-satiated, $\chi = R^L_+$, $p >> 0$, $w > 0$ and $u > u(0)$, then*

1. $x^* \in \xi(p,w)$ *implies* $x^* \in \eta(p, v(p,w))$ *and* $e(p, v(p,w)) = w$
2. $x^* \in \eta(p,u)$ *implies* $x^* \in \xi(p, e(p,u))$ *and* $v(p, e(p,u)) = u$

Proof ? [3] page 58

Theorem 2.21 *The Slutsky Equation: If $u(x) \in C^0$, $\succeq$ is locally non-satiated and strictly convex, $\chi = R^L_+$, and $u = v(p,w)$ then $D_p h(p,u) = D_p x(p,w) + D_w x(p,w) \cdot x(p,w)$.*

Proof By duality, $h(p,u) = x(p,e(p,u))$ and $e(p,u) = w$. So $D_p h(p,u) = D_p x(p,e(p,u)) + D_w x(p,e(p,u)) \nabla_p e(p,u) = D_p x(p,e(p,u)) + D_w x(p,e(p,u)) h(p,u) = D_p x(p,e(p,u)) + D_w x(p,e(p,u)) x(p,e(p,u))$. [3] page 71 ∎

2.6 Integrability

section 3.h [3]

2.7 Individual Welfare

Equivalent Variation (EV) and Compensating Variation (CV) are measures of the welfare effects of a change in prices.

Definition For a price change, p^0 to p^1, $v(p^0, w + EV[p^0, p^1]) = u^1$.

Remark EV is the wealth change required to attain the new utility level (after the price change) at the old price level. For a positive price change, this is the amount that would need to be paid to avoid the price change.

Definition For a price change, p^0 to p^1, $v(p^1, w - CV[p^0, p^1]) = u^0$.

Remark CV is the wealth change required to attain the old utility level (after the price change) at the new price level. For a positive price change, this is the amount the agent would be willing to pay to make the price change happen.

Definition Deadweight loss: Let $p^0 = (p_1^0, \bar{p}_{-L})$ and $p^1 = (p_1^0 + t, \bar{p}_{-L})$, where t is a tax on the first good. Let $T = t \cdot x_1(p^1, w)$, then $DL = -T - EV(p^0, p^1)$.

Remark Deadweight loss is the difference between a price change on a single good and a wealth transfer of the same size.

i think exercises for this section 3.i in mas-col

3 Generalizations

3.1 Jevonian demand and Labor Economics

Definition Jevonian wealth is a function of a price variant endowment (ω), dividends from firm profits ($\Pi(p)$) and a price invariant endowment; $w(p, m) = p \cdot \omega + \Pi(p) + m$.

Theorem 3.1 *Slutsky equation for Jevonian wealth: For* $J(p, m) = x(p, w(p, m))$, $D_pJ(p, m) = D_ph(p, u) - D_wx(p, w(p, m))(x(p, w(p, m)) - \omega - \nabla\pi(p))$.

3.2 Uncertainty

Here preferences are over a set of lotteries, $\mathcal{L}$, which are CDFs over χ the set of possible outcomes. We'll take $\chi = R_+^L$ as usual.

Definition A *simple lottery* $L \in \mathcal{L}$ is a list $L = (p_1 \cdots p_N), p_i \geq 0, \sum p_i = 1$.

Definition A *compound lottery*, given K simple lotteries $L_k = (p_1^k \cdots p_N^k)$ and $K\alpha_k \geq 0$ s.t. $\sum \alpha_k = 1$, is the risky alternative $(L_1, \cdots, L_K; \alpha_1 \cdots \alpha_K)$ that yields the simple lottery L_k with probability $\alpha_k, \forall k$.

Definition Preferences $\succeq$ over $\mathcal{L}$ satisfies the *independence axiom* if $\forall L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$, $L \succeq L'$ if and only if $\alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L''$.

Definition Preferences $\succeq$ over $\mathcal{L}$ satisfy the *expected utility hypothesis*, if they can be represented by a utility function, $U : \mathcal{L} \to R$, that can be written as an expectation, i.e. $U(L) = E_L u(x)$.

Remark Preferences can be represented by a multitude of utility functions. The above definition only says that *one* of those representations must be of the expectation form for the hypothesis to be satisfied.

Result 3.2 *Expected utility theorem: If preferences $\succeq$ are rational and they satisfy the independence axiom then $\succeq$ satisfy the expected utility hypothesis, i.e. $\exists u : \chi \to R$ s.t. $U : \mathcal{L} \to R : U(L) = E_L u(x)$ represents $\succeq$.*

Definition *Jensen's inequality*: $\succeq$ (and expected utility hyp is met) display *risk aversion* if $\forall L \in \mathcal{L}$, $E_L u(x) \leq u(E_L x)$.

Remark ARA, RRA... CRRA

Example Insurance

4 Aggregate Demand

Definition The positive representative consumer: Let $\bar{x} : R_{++}^L \times R_+^I \to \chi, \bar{x}(p,w) = \sum_{i=1}^I \tilde{x}^i(p, w_i)$. A representative agent has a solution to his UMAX problem $(\hat{x})$ such that $\hat{x}(p, \sum w_i) = \bar{x}(p,w)$.

Definition Aggregate demand is independent of the distribution of wealth if $\forall i, j, \sum w_i^k = \sum w_i^j$ then $\sum \tilde{x}^i(p, w_i^0) = \sum \tilde{x}^i(p, w_i^1)$.

Lemma 4.1 *Aggregate demand is independent of the distribution of wealth if and only if the Engel curves of every consumer are affine and have the same slope (i.e. demand is of the gorman form1.5 with* $\forall i, k \beta_j^i(p) = \beta_j^k(p)$*).*

Proof If $x_j^i(p, w_i) = \alpha_j^i(p) + \beta_j(p) w_i$ then $\sum_i x_j^i(p, w_i) = \sum_i \alpha_j^i(p) + \beta_j(p) \sum_i w_i$ so whatever the distribution of wealth, aggregate demand won't depend on it.

If $x_j^i(p, w_i - \epsilon) + x_j^h(p, w_i + \epsilon) + \sum_{k \neq i,j} x_j^k(p, w_k) = \sum_k x_j^k(p, w_k)$ then $-\frac{\partial x_j^i}{\partial w} + \frac{\partial x_j^h}{\partial w} = 0$. Thus, the Engel curves (i.e. the demand of good j as a function of wealth) have the same slope. ■

Lemma 4.2 *If* $\forall i$*, indirect utility can be expressed as a function in the gorman form 1.5, aggregate demand is independent of the distribution of wealth.*

Proof By Roy's identity 2.14, $\tilde{x}_j^i(p, w_i) = \frac{\frac{\partial v}{\partial p_j}}{\frac{\partial v}{\partial w}} = \frac{\frac{\partial \alpha_j^i}{\partial p_j} + \frac{\partial \beta_j}{\partial p_j} w_i}{\beta_j(p)}$. So, $\sum_i \tilde{x}_j^i(p, w_i) = \frac{\sum_i \frac{\partial \alpha_j^i}{\partial p_j} + \frac{\partial \beta_j}{\partial p_j} \sum_i w_i}{\beta_j(p)}$

Result 4.3 *existence of rep consumer (11/13/05), gorman positive*

Result 4.4 *eisenburg positive*

5 Production

The profit maximization problem given prices p and production set Y,

$$\max_{y \in Y} p \cdot y \tag{10}$$

The cost minimization problem given output level q, input prices w, and production function $f(z)$,

$$\max_{f(z) \geq q} w \cdot z \tag{11}$$

Definition *Production set*: $Y = \{y \in R^L : F(y) \leq 0\}$

Definition *Production function* (in the case of only one output): $f : R_+^{L-1} \to R_+, f(z) = q$

Definition Common Assumptions:

1. "Free disposal" $Y - R_+^L \subset Y$
2. $Y \neq \emptyset$
3. Y is closed
4. "No free lunch" $Y \cap R_+^L \subset \{0\}$
5. "Possibility of inaction" $0 \in Y$
6. "Non-increasing returns to scale" (NIRS) $y \in Y, \alpha \in [0,1] \Rightarrow \alpha y \in Y$
7. "Non-decreasing returns to scale" (NDRS) $y \in Y, \alpha > 1 \Rightarrow \alpha y \in Y$
8. "Constant returns to scale" (CRS) $y \in Y, \alpha \geq 0 \Rightarrow \alpha y \in Y$
9. "Convexity" Y is convex
10. "Additivity" $y^0, y^1 \in Y \Rightarrow y^0 + y^1 \in Y$

Result 5.1 *Convexity and possibility of inaction imply NIRS*

Result 5.2 *Additivity and NIRS imply CRS*

Result 5.3 *Let ν be the solution correspondence ("supply") (and $\tilde{y}$ the solution function if the correspondence is single valued) to the profit max problem 10 and π be its value function ("profit"). Assume Y is closed and there's free disposal, then:*

1. *π is 1-homo*
2. *π is a convex function*
3. *ν is 0-homo*
4. *Y convex $\Rightarrow \nu$ is convex*
5. *If π is diff'ble at p then $\nabla \pi(p) = \tilde{y}(p)$*

6. *If $\tilde{y}$ is diff'ble at p then $D\tilde{y} = D^2\pi(p)$ is a positive semi-definite and symmetric matrix where $D\tilde{y}p = 0$.*

7. *Law of supply, $\Delta y \cdot \Delta p \geq 0$*

Remark The law of supply can be derived as such... $(p' - p) \cdot (y' - y) = (p'y' - py') + (py - p'y) \geq 0$ which follows from the fact that y maximizes profits for prices p and y' maximizes profits for prices p'.

Remark See M-C [3] pages 143-147 for discussion on convexities and supply loci

Definition Given J firms, the *aggregate production set* is $Y = \sum Y^j$.

Definition $y \in Y$ is *efficient* if there doesn't exist a $y' \in Y$ s.t. $y' \geq y$ and $y \neq y'$.

Theorem 5.4 *If $y \in Y$ is profit maximizing then it is efficient.*

Proof Suppose $\exists y' \neq y \in Y$ s.t. $y' \geq y$ then because prices are strictly positive $py' > py$. But if $y \in Y$ is profit maximizing then $\forall y' \neq y \in Y, py \geq py'$. The contradiction implies no such y' exists. ∎

Theorem 5.5 *If Y is convex and $y \in Y$ is efficient then $\exists p \neq 0$ s.t. py is profit maximizing.*

Proof Sketch: A hyperplane (i.e. a set of non-zero prices p) separates the convex Y from the convex set $P_y = \{y\} + R_{++}$ because $Y \cap P_y = \emptyset$. In particular, $py' \geq py, \forall y' \in P_y$. Now, for any particular $y'' \in Y$, $py' \geq py'', \forall y' \in P_y$, but y' can be chosen arbitrarily close to y so $py \geq py'', \forall y'' \in Y$, i.e. y is profit maximizing.

Remark The above two theorems are the first instance of the fundamental theorems of welfare economics.

6 General Equilibrium

Definition Let s', s'' be feasible states of the world. s' is *Pareto superior* to s'' if

1. $s' \succeq^i s'', \forall i$
2. $\exists i$ s.t. $s' \succ^i s''$

-kaldor criteria

6.1 Exchange economies

Edgeworth Box: know it, love it

Definition A *competitive equilibrium* of an exchange economy $\xi(u, \omega)$, where u are the preferences of the I agents represented as utility functions and ω are the initial endowments of the L goods, is a pair $(x, p) \in R_+^{LI} \times R_+^L$ s.t. $\forall i, x^i \in \arg\max\{u^i(x^i) : px^i \leq p\omega^i, x^i \geq 0\}$ (agents maximize their budget constrained utility) and $\sum x^i = \sum \omega^i$ (markets clear).

Result 6.1 *Quantity feasibility implies value feasibility*

Lemma 6.2 *If preferences are locally non-satiated, (x,p) is a competitive equilibrium and* $u(x') \geq u(x)$ *for some agent, then* $p \cdot x' \geq p \cdot x$

Proof Assume $p \cdot x' < p \cdot x$. Because of LNS, $\forall \epsilon, \exists x'' \in B_\epsilon(x')$ s.t. $u(x'') > u(x') \geq u(x)$. Because x'' is arbitrarily close to x', in the limit and by continuity $p \cdot x'' < p \cdot x$. But this contradicts utility maximization (x'' gives more utility and its feasible). Thus, $p \cdot x' \geq p \cdot x$. ▮

Theorem 6.3 *First Fundamental Theorem of Welfare Economics in an Exchange Economy: If all agents have locally non-satiated preferences then any competitive equilibrium* (x, p) *is Pareto optimal.*

Proof Assume not, then $\exists x_0$ s.t. $\forall i, u^i(x_0^i) \geq u^i(x^i)$ and for some agent $j, u^j(x_0^j) > u^j(x^j)$. This implies that $\forall i, px_0^i \geq p\omega^i$ and for $j, px_0^j > p\omega^j$. Summing this implies, $\sum px_0^i > \sum p\omega^i$ but if x_0 is feasible $\sum x_0^i = \sum \omega^i$ then $\sum px_0^i = \sum p\omega^i$. We have a contradiction so no such x_0 exists and the equilibrium is P.O. ▮

Definition (x, p) is a *quasi-equilibrium* if $\forall i, u^i(x_0^i) \geq u^i(x^i) \Rightarrow px_0^i \geq px^i$ (agents are minimizing costs) and $\sum x_i = \sum \omega^i$ (markets clear).

Remark A quasi-equilibrium is an equilibrium if for all agents $h(p, u^i) = \tilde{x}^i(p, e(p, u^i))$, i.e. the case when there's duality between the preference maximization and expenditure minimization.

Result 6.4 *Second Fundamental Theorem of Welfare Economics in an Exchange Economy: If preferences are convex and locally non-satiated then* $\forall x$ *that is P.O.,* $\exists p$ *s.t.* (x, p) *is a quasi-equilibrium.*

6.2 Production economies

Definition A *competitive equilibrium* in production economy with private property $\xi(u,\omega,Y,\theta)$; where u are the I agents preferences, ω are their endowments, Y are the J firms' productions sets and $\sum_i \theta_j^j = 1, \forall j$ represents the ownership of the firms by the agents; is a triple (x,y,p) s.t.

1. $x^i \in \arg\max\{u^i(x^i) : p \cdot x^i \leq p \cdot \omega^i + \theta^i \cdot \Pi(p), x^i \geq 0\}$
2. $y^j \in \arg\max\{\Pi^j(p) = p \cdot y^j : y^j \in Y^j\}$
3. $\sum x^i = \sum \omega^i + \sum y^j$

Result 6.5 *First Fundamental Theorem of Welfare Economics in a Production Economy with Private Property: If all agents in the economy $\xi(u,\omega,Y,\theta)$ have locally non-satiated preferences then any competitive equilibrium (x,y,p) is Pareto optimal.*

Theorem 6.6 *Second Fundamental Theorem of Welfare Economics in a Production Economy with Private Property: In a private property economy $\xi(u,\omega,Y,\theta)$ s.t.*

(U) $\forall i, u^i$ is continuous, quasi-concave and satisfy local non-satiation

(P) $\forall j, Y^j$ is convex and satisfies free disposal

if (x,y) is Pareto optimal then $\exists p \neq 0 \in R_+^L$ s.t. (x,y,p) is a quasi-equilibrium.

6.3 Differentiable utility

6.4 Existence

7 Market Failure

7.1 Externalities

In this section, assuming interior solutions only.[2]

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[2] For this and the next section see [2]

Given two firms and a consumer, where the consumption and production of the good that firm 1 produces impacts the production of the second good. Central planner's problem:

$$\max_{x_1,x_2,y_1^1,y_1^2,y_2^1,y_2^2} u_(x_1, x_2) \qquad s.to \tag{12}$$

$$\begin{aligned} x_1 &\leq y_1^1 + y_1^2 + \omega^1 && (13)\\ x_2 &\leq y_2^1 + y_2^2 + \omega^2 && (14)\\ y_1^1 &\leq f^1(y_2^1) && (15)\\ y_2^2 &\leq f^2(y_1^2, y_1^1, x_1) && (16) \end{aligned}$$

has FOC:

$$\frac{\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial f^2}{\partial x_1}}{\partial u/\partial x_2} = -\frac{\partial f^2}{\partial y_1^2} = -\frac{1 + \frac{\partial f^2}{\partial y_1^1} \cdot \frac{df^1}{dy_2^1}}{df^1/dy_2^1} \tag{17}$$

which equalizes the social marginal rates of substitution and transformation. It can be seen that the private marginal rates of substitution and transformation aren't going to be equal to the social values for the consumer and first firm. Thus, the equilibrium is not efficient.

Proposition 7.1 *Coase Theorem: If property rights are assigned to an externality, then bargaining will lead to an efficient outcome no matter how those rights were assigned.*

Assume firm 1 has to buy from firm 2 the rights to pollute while producing good 1 and the consumer, likewise, has to buy the rights from firm 2 to pollute while consuming good 1. Firm 1's profit maximization problem is:

$$\max_{y_1^1,y_2^1} (p_1 - e_1)y_1^1 + p_2 y_2^1 \qquad s.to \tag{18}$$

$$y_1^1 \leq f^1(y_2^1) \tag{19}$$

with FOC:

$$\frac{df^1}{dy_2^1} = \frac{p_2}{e_1 - p_1} \tag{20}$$

Firm 2 has problem, with y_1^1 and x_1 begin the costless pollution rights sold by the firm:

$$\max_{y_1^2,y_2^2,y_1^1,x_1} p_1 y_1^2 + p_2 y_2^2 + e_1 y_1^1 + e_2 x_1 \qquad s.to \tag{21}$$

$$y_2^2 \leq f^2(y_1^2, y_1^1, x_1) \tag{22}$$

with FOC:

$$\frac{p_1}{p_2} = \frac{\partial f^2}{\partial y_1^2} \tag{23}$$

$$\frac{e_1}{p_2} = \frac{\partial f^2}{\partial y_1^1} \tag{24}$$

$$\frac{e_2}{p_2} = \frac{\partial f^2}{\partial x_1} \tag{25}$$

and the consumer has utility maximization problem:

$$\max_{x_1,x_2} u(x_1, x_2) \qquad s.to \tag{26}$$

$$p_1x_1 + p_2x_2 \leq p_1\omega_1 + p_2\omega_2 - e_2x_1 \tag{27}$$

gives FOC:

$$\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{p_1 + e_2}{p_2} \tag{28}$$

These FOC are, taken together, equivalent to the social planner's FOC. Thus, this arrangement leads to the equilibrium being optimal.

Alternatively, a Pigouvian tax could be enacted that taxes externalizing behavior and transfers the tax revenue as a lump-sum to the consumer. The consumer sees a lump-sum and they don't appreciate that their actions can change that amount. Thus, an equilibrium is such that:

1. Firm 1 maximizes $(p_1-\tau)y_1^1+p_2y_2^1$ subject to their technology constraint
2. Firm 2 maximizes $p_1y_1^2 + p_2y_2^2$ subject to their technology constraint
3. The consumer maximizes utility subject to $(p_1 + t)x_1 + p_2x_2 \leq p_1\omega_1 + p_2\omega_2 + \Pi(p) + T$
4. $T = t + \tau, \Pi(p) = (p_1 - \tau)y_1^1 + p_2y_2^1 + p_1y_1^2 + p_2y_2^2, y_1^1 + y_1^2 + x_1 = \omega_1, y_2^1 + y_2^2 + x_2 = \omega_2$

It can be shown that with prices $(\frac{\partial u/\partial x_1}{\partial u/\partial x_2} + \frac{\partial f^2}{\partial x_1}, 1)$ and taxes $\tau = -\frac{\partial f^2}{\partial y_1^1}$ and $t = -\frac{\partial f^2}{\partial x_1}$ the corresponding equilibrium allocation is Pareto optimal.

Remark pollution abatement

7.2 Public Goods

Central planner's problem with public good $y = g(z)$ (made from the private good), private good x and $\alpha_i \geq 0$:

$$\begin{aligned}
\max_{x,y} \sum \alpha_i u_i(x^i, y) & \qquad s.to \qquad (29)\\
\sum x^i + z &\leq \sum \omega^i \qquad (30)\\
y &\leq g(z) \qquad (31)\\
-x^i &\leq 0, \forall i \qquad (32)\\
-z &\leq 0 \qquad (33)\\
-y &\leq 0 \qquad (34)
\end{aligned}$$

has first order necessary conditions for an interior solution:

$$\sum \frac{\partial u^i/\partial y}{\partial u^i/\partial x^i} = \frac{1}{g'} \qquad (35)$$

Voluntary contribution equilibrium: Assuming each agent contributes z^i of the private good for the production of the public good, then $y = g(\sum z^i)$. Then each of the consumers has the following problem:

$$\begin{aligned}
\max_{x^i,z^i} u_i(\omega^i - z^i, g(\sum z^k)) & \qquad s.to \qquad (36)\\
-x^i &\leq 0 \qquad (37)\\
-z^i &\leq 0 \qquad (38)
\end{aligned}$$

which has FOC of

$$\frac{\partial u^i/\partial y}{\partial u^i/\partial x^i} = \frac{1}{g'} \qquad (39)$$

clearly summing these FOC shows that voluntary contribution isn't Pareto optimal.

Lindahl equilibrium: Each agent has their own price for the public good p_i and solves the following problem:

$$\begin{aligned}
\max_{x^i,y} u_i(x^i, y)) & \qquad s.to \qquad (40)\\
x^i + p_i y &\leq \omega_i \qquad (41)\\
-x^i &\leq 0 \qquad (42)\\
-z^i &\leq 0 \qquad (43)
\end{aligned}$$

gives us FOC $\frac{\partial u^i/\partial y}{\partial u^i/\partial x^i} = p^i, \forall i$ and demand for $x^i(p^i)$ and $y^i(p^i)$. A firm produces the public good facing the price $p = \sum p^i$ and must solve the following profit maximization problem:

$$\begin{aligned}\max_z py - z \qquad & s.to & (44)\\ y \;\leq\; & g(z) & (45)\\ -y \;\leq\; & 0 & (46)\\ -z \;\leq\; & 0 & (47)\end{aligned}$$

which gives FOC $p = g'^{-1}$, demand of input $z(p)$ and a supply of $y(p)$. Then an equilibrium is a set of prices $(p_1, \cdots, p_I)$ s.t. $\sum p^i = p$, $y(p) = y^i(p^i), \forall i$ and $\sum x^i + z(p) = \sum \omega^i$.

Combining the FOC we get $\sum \frac{\partial u^i/\partial y}{\partial u^i/\partial x^i} = \sum p^i = p = g'^{-1}$. Thus, the Lindahl equilibrium is Pareto optimal.

- equilibrium with tax and majority voting, $u((1-t)\omega^i, g(t\sum \omega^i))$.

7.3 Asymmetrical Information

Definition *Adverse selection* refers to a situation with asymmetrical information in which two goods with different characteristics (one good is 'better' than the other) can be distinguished by one party but not another. This asymmetry removes the market for the better good and leaves only the worse good.

Definition *Signals* are chosen by the agent with private knowledge to indicate some unobserved quality.

Definition *Indexes* are not chosen by the agent with private knowledge but nevertheless it indicates an unobserved quality.

Definition The Pivotal Mechanism:

1. Each agent pays a set amount and $\sum c_i = C$ the total cost of the project
2. Each agent states their gross benefit (call player i's ω_i), this may be different from their true value v_i, but v_i is kept as private information.
3. Carry out project if $\sum \omega_i > C$

4. Agent i is pivotal if $\sum \omega_j > C$ but $\sum \omega_j - \omega_i \leq C - c_i$ or $\sum \omega_j \leq C$ but $\sum \omega_j - \omega_i > C - c_i$
5. If agent i is pivotal then she pays tax equal to $|\sum_{j \neq i} \omega_j - \sum_{j \neq i} c_j|$ whether or not the project is carried out

Result 7.2 $\forall i, \omega_i = v_i$ *is a weakly dominant strategy in the pivotal mechanism game.*

7.4 Monopoly and Oligopoly

Definition Bertrand competition has n firms who choose the price for a single homogeneous good.

Definition Cournot competition has n firms who choose the quantity a single homogeneous good to produce, i.e. $q_i \in S_i = R_+$. The firms face a demand function of the form $p = p(Q)$ where Q is the total production of the good.

Lemma 7.3 *As you add more identical firms, Cournot competition, with linear demand and cost functions, is perfect competition in the limit.*

Proof FOC's for a Cournot firm is $\frac{\partial \Pi_i}{\partial q_i} = p(q) + q_i \frac{\partial p}{\partial q_i} - \frac{\partial c_i}{\partial q_i} = 0$. Identical firms with linear demand $(a + bq)$ and cost functions $(d + cq))$ imply each firm produces $q^* = \frac{a-c}{b(n+1)}$ and total production is $Q^* = \frac{n(a-c)}{b(n+1)}$. This implies an equilibrium price with n firms of $p_n^* = \frac{a+nc}{n+1} = \frac{\frac{a}{n}+c}{1+\frac{1}{n}}$. Take the limit to get the result. ∎

Remark Here's my stab at proving the general case, without the assumption of linearity. From above, $p(q) = \frac{\partial c_i}{\partial q_i} - q_i \frac{\partial p}{\partial q_i}$. Summing across firms, $np(q) = \sum \frac{\partial c_i}{\partial q_i} - \sum q_i \frac{\partial p}{\partial q_i}$. Because consumers are indifferent to which firm they purchase the good from, the total expenditure will be the same in equilibrium. This implies the very right hand term goes to zero in a Walras' Law, Engel's aggregation sort of way. Now, we have $p(q) = 1/n \sum \frac{\partial c_i}{\partial q_i}$, the average marginal cost.

7.4.1 Price discrimination

Definition *First degree price discrimination*: Firm faces different consumers with different demand functions and can discriminate perfectly between them. All consumer surplus is extracted, usually via a two-tier tariff pricing scheme (set price equal to marginal cost but charge a fixed fee equal to the consumer surplus).

Proposition 7.4 *First degree price discrimination is efficient.*

Definition *Second degree price discrimination*: Firm faces different groups of consumers with different aggregate demand functions and isn't able to discriminate between them. The firm offers a menu of packages intended to induce the consumers to self-select into the appropriate groups.

Example With 2 types of consumers, the firm can offer a menu of just one package (Q_l, V_l) intended to attract all consumers (and taking all of the l-type consumer's surplus), a menu of just one package (Q_h, V_h) intended to attract just one of the types of consumers (and taking all of the h-type consumer's surplus) or a menu of two packages $((Q_h, V_h), (Q_l, V_l))$.

In the third option, each type of consumer will have to self-select the package that was intended for them. This means, letting $W_i(Q_j)$ be the willingness of consumer type i to pay for Q_j of the good:

1. $W_h(Q_h) \geq V_h$
2. $W_l(Q_l) \geq V_l$
3. $W_h(Q_h) - V_h \geq W_h(Q_l) - V_l$
4. $W_l(Q_l) - V_l \geq W_l(Q_h) - V_h$

By assumption, $W_h(q) > W_l(q), \forall q$, so #2 implies, with #3, $W_h(Q_h) - V_h > 0$. So, #1 is implicit in the other items. #3 is satisfied no matter how large V_l becomes, thus #2 and #3 will be satisfied with equality. So, $V_h = W_h(Q_h) - W_h(Q_l) + W_l(Q_l)$ and $V_l = W_l(Q_l)$.

Now choose Q_h and Q_l to maximize profit (check the Hessian of profit to make sure we're finding a max).

Definition *Third degree price discrimination*: Firm faces different groups of consumers with different aggregate demand functions but can discriminate between the groups. The firm will just treat each type of consumer as a different market.

7.5 Interactive Knowledge

8 Greek Letters

Uppercase	*Lowercase*	*name*
	α	*alpha*
	β	*beta*
	χ	*chi*
Δ	δ	*delta*
	ϵ	*epsilon*
	η	*eta*
Γ	γ	*gamma*
	ι	*iota*
	κ	*kappa*
Λ	λ	*lambda*
	μ	*mu*
	ν	*nu*
Ω	ω	*omega*
Φ	ϕ	*phi*
Π	π	*pi*
Ψ	ψ	*psi*
	ρ	*rho*
Σ	σ	*sigma*
	τ	*tau*
Θ	θ	*theta*
Υ	υ	*upsilon*
Ξ	ξ	*xi*
	ζ	*zeta*

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